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Spin-Polarized ^{3}He–HeII Mixtures in the Static Fluctuation Approximation

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We have used the so-called static fluctuation approximation (SFA) to calculate the thermodynamic properties of spin-polarized ^{3}He–HeII mixtures at low temperature, $T < 0.025\,\mathrm{K}$. This approximation is based on the replacement of the square of the local-field operator with its mean value. A closed set of nonlinear integral equations is derived for spin-up and spin-down systems. This set is solved numerically by an iteration method for a realistic interhelium potential. The mean internal energy per unit volume, the pressure, the entropy per unit volume, and the specific heat per unit volume increase with increasing temperature. The mean internal energy per unit volume, the pressure increase with increasing spin polarization; while the entropy per unit volume and the specific heat per unit volume are weakly-dependent on spin polarization.

KEY WORDS: spin-polarized system; static fluctuation approximation; ^{3}He-HeII mixtures.

1. INTRODUCTION

In this paper, we shall study the thermodynamic properties of the spin-polarized ^{3}He–HeII mixtures from the microscopic point of view. This is a weakly-interacting neutral many-fermionic system.

The ^{3}He–HeII mixture is an interesting system for several reasons. For one, it is widely used as a refrigerant for cooling purposes up to the milliKelvin range (Kelly, 1973). For another, it is a model system for testing theories of weakly-interacting fermions. The polarized mixture provides us with an even better testing ground for the underlying quantum physics. It is a 'natural laboratory' for studying properties of matter in an extremely pure environment, where most of the unavoidable disturbances present at higher temperatures are almost completely 'frozen out.'

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This system has been studied theoretically from various perspectives. The magnetokinetic effects have been investigated at arbitrary temperatures, impurity concentrations, and magnetic fields (Meyerovich, 1978). In strong magnetic fields, the kinetic coefficients have been found to increase exponentially with the field. Also, the superfliud phases of ^{3}He in ^{3}He–HeII mixtures in the presence of a magnetic field have been discussed (Meyerovich, 1980). Observation of these phases seems to be possible, and their properties appear to be as interesting as the properties of superfluid (pure) ^{3}He. A variational method has been used to determine the viscosity of the dilute spin-polarized mixture (Hampson *et al.*, 1988). Variations in the thermodynamic properties arise from the changes in both the density of states and the kinetic energy of the quasiparticles (Bradley, 1997).

Experimentally, there have been two major tracks. The first has been the study of the spin dynamics and transport properties of these mixtures. Pulsed-NMR techniques have been used to investigate longitudinal spin diffusion and nonlinear spin dynamics in dilute spin-polarized ^{3}He–HeII mixtures between 4 and 400 mK (Nunes *et al.*, 1992). Transverse spin diffusion and spin rotation in very dilute spin-polarized ^{3}He–HeII mixtures have been measured by an 8-T magnetic field<TB:break/> (Candela *et al.*, 1991). The second track has been the development of the technology to produce spin-polarized mixtures. A polarization of 56% has been obtained in a 4% ^{3}He–HeII mixture at 200 mK (Candela *et al.*, 1994). Spin-polarized ^{3}He–HeII mixtures are prepared by the fast liquefaction of a vapor highly polarized by laser optical pumping. A nuclear polarization of order 25% has been obtained in the liquid (Villard *et al.*, 2000).

In this paper, we are interested in the thermodynamic properties of spin-polarized ^{3}He–HeII mixtures at temperatures less than the Fermi degeneracy temperature $\sim$0.1 K. We shall apply for the first time in this context the so-called static fluctuation approximation (SFA). The SFA has already been used to study various systems, such as the classical two-dimensional Ising model (Nigmatullin *et al.*, 2000a); liquid helium-4 (Al-Sugheir *et al.*, 2001); liquid helium-3 (Al-Sugheir and Ghassib, 2002); and ^{3}He–HeII (Al-Sugheir, 2004). The underlying physical idea is to replace the square of the local-field operator with its mean value. The physical implication is that the true quantum-mechanical spectrum of this operator is replaced with a distribution around the expectation value of the local-field operator (Al-Sugheir, 2004; Al-Sugheir *et al.*, 2001; Al-Sugheir and Ghassib, 2002; Nigmatullin *et al.*, 2000a,b,c; Nigmatullin and Toboev, 1989).

We shall consider an extended system of N_3 ^{3}He atoms, each of mass m_3, occupying a volume Ω_3, embedded in a HeII background of N_4 atoms, occupying a volume Ω_4. We know that ^{4}He atoms are spinless bosons, while ^{3}He are fermions with spin 1/2. It should be noted that this mixture of ^{3}He–HeII, at temperatures less than the Fermi degeneracy temperature ($\sim$0.1 K at zero pressure), is a dilute weakly-interacting neutral *many-fermionic* system. This is because of the negligible number density of Bose-type excitations (phonons and rotons) under these circumstances, and the dominance of the ^{3}He quasiparticles (Kittel and Kroemer, 1995; Wilks, 1967). ^{3}He–HeII mixtures have an additional degree of freedom, which is the ^{3}He concentration. This degree of freedom enables us to study the density effect on the various properties.

The rest of the paper is organized as follows. Section 2 involves a full derivation of the closed set of nonlinear integral equations for spin-polarized ^{3}He–HeII mixtures. Section 3 is devoted to the calculations and numerics. Section 4 summarizes the results. Finally, in Section 5, the paper closes with some concluding remarks.

2. THE STATIC FLUCTUATION APPROXIMATION FOR SPIN-POLARIZED ^{3}He–HeII MIXTURES

For a specific Hamiltonian $\hat{H}$, the Heisenberg representation of a creation operator $\hat{a}^+_{k\lambda}$ is given by

$$\hat{a}^+_{k\lambda}(\tau) = \exp(\tau \hat{H})\hat{a}^+_{k\lambda}(0)\exp(-\tau \hat{H}), \tag{1}$$

where $\tau \equiv it$. The equation of motion of the creation operator in this representation can be written in the form

$$\frac{d\hat{a}^+_{k\lambda}(\tau)}{d\tau} = [\hat{H}, \hat{a}^+_{k\lambda}(\tau)]. \tag{2}$$

Here k and λ are indices denoting the complete set of compatible quantum numbers describing a specific state, where $\vec{k}$ denotes the linear momentum of the particle and $\lambda = \pm\frac{1}{2}$ is the spin of the particle. According to the well-known anticommutation relations for a Fermi system:

$$\{\hat{a}_{k\lambda}, \hat{a}^+_{q\lambda_1}\} = \delta_{kq}\delta_{\lambda\lambda_1}; \quad \{\hat{a}_{k\lambda}, \hat{a}_{q\lambda_1}\} = \{\hat{a}^+_{k\lambda}, \hat{a}^+_{q\lambda_1}\} = 0.$$

The total Hamiltonian for neutral many-fermionic systems in second quantization can be written as

$$\begin{aligned}\hat{H} &= \int d\vec{r}\hat{\Psi}^+(\vec{r})\left(-\frac{\hbar^2}{2m}\nabla^2\right)\hat{\Psi}(\vec{r}) \\ &\quad + \frac{1}{2}\int\int d\vec{r}_1 d\vec{r}_2 \hat{\Psi}^+(\vec{r}_1)\hat{\Psi}^+(\vec{r}_2)V(\vec{r}_1 - \vec{r}_2)\hat{\Psi}(\vec{r}_2)\hat{\Psi}(r_1).\end{aligned} \tag{3}$$

Here $\hat{\Psi}(\vec{r})$ and $\hat{\Psi}^+(\vec{r})$ are the field operators, $\hbar$ is Planck's constant ($\hbar \equiv h/2\pi \equiv$ Dirac's constant), m is the fermionic mass, and $V(\vec{r}_1 - \vec{r}_2)$ is the pairwise central potential that depends only on the modulus of $|\vec{r}_1 - \vec{r}_2|$.

It is convenient to write the field operators as linear combinations of the creation and annihilation operators:

$$\hat{\Psi}(\vec{r}) = \sum_{\vec{k}\lambda} \psi_{\vec{k}\lambda}(\vec{r})\hat{a}_{\vec{k}\lambda}; \tag{4}$$

$$\hat{\Psi}^{+}(\vec{r}) = \sum_{\vec{k}\lambda} \psi^{+}_{\vec{k}\lambda}(\vec{r})\hat{a}^{+}_{\vec{k}\lambda}, \tag{5}$$

where the coefficients $\psi_{\vec{k}\lambda}(\vec{r})$, $\psi^{+}_{\vec{k}\lambda}(\vec{r})$ are the single-particle wavefunctions and the sum is over the complete set of single-particle quantum numbers.

In a uniform homogeneous infinite system, all physical properties must, of course, be invariant under spatial translations. This suggests that the single-particle wavefunctions are plane waves:

$$\psi_{\vec{k}\lambda}(\vec{r}) = \frac{1}{\sqrt{\Omega}} \exp(i\vec{k}\cdot\vec{r})\eta_\lambda, \tag{6}$$

where Ω is the normalization volume of the system and η_λ are the two spin wavefunctions.

Integration over the spatial coordinates of Eq. (3) gives the Hamiltonian in second quantization (Al-Sugheir, 2004; Fetter and Walecka, 1971; Stoof *et al.*, 1996):

$$\hat{H} = \sum_{\vec{k}\lambda} \varepsilon(\vec{k})\hat{a}^{+}_{\vec{k}\lambda}\hat{a}_{\vec{k}\lambda} + \frac{1}{2\Omega}\sum_{\vec{k}\vec{p}\vec{q}}\sum_{\lambda\lambda_1} V(q)\hat{a}^{+}_{\vec{k}+\vec{q},\lambda}\hat{a}^{+}_{\vec{p}-\vec{q},\lambda_1}\hat{a}_{\vec{p}\lambda_1}\hat{a}_{\vec{k}\lambda}, \tag{7}$$

where $\varepsilon(\vec{k})$ is the kinetic energy:

$$\varepsilon(\vec{k}) = \frac{\hbar^2 k^2}{2m_3^*}, \tag{8}$$

m_3^* being the effective mass of a ^{3}He atom, and $V(q)$ being the Fourier transform of the pair potential defined as

$$V(q) = \int V(r)\exp(i\vec{q}\cdot\vec{r})d\vec{r}. \tag{9}$$

The grand canonical Hamiltonian of this system is

$$\hat{H} = \sum_{\vec{k}\lambda} (\varepsilon(\vec{k}) - \mu_\lambda)\hat{a}^{+}_{\vec{k}\lambda}\hat{a}_{\vec{k}\lambda} + \frac{1}{2\Omega}\sum_{\vec{k}\vec{p}\vec{q}}\sum_{\lambda\lambda_1} V(q)\hat{a}^{+}_{\vec{k}+\vec{q},\lambda}\hat{a}^{+}_{\vec{p}-\vec{q},\lambda_1}\hat{a}_{\vec{p}\lambda_1}\hat{a}_{\vec{k}\lambda}, \tag{10}$$

where μ_λ is the effective chemical potential, which depends on the spin state of the particle for a spin-polarized system. For spin-up particles:

$$\begin{aligned} \mu_\uparrow &= \mu_3 + \vec{M}_{\text{He}}\cdot\vec{B} \\ &= \mu_3 + M_{\text{He}}B; \end{aligned} \tag{11}$$

and for spin-down particles:

$$\begin{aligned}\mu_\downarrow &= \mu_3 + \vec{M}_{\mathrm{He}} \cdot \vec{B} \\ &= \mu_3 - M_{\mathrm{He}} B,\end{aligned} \tag{12}$$

where μ_3 is the chemical potential of ^{3}He–HeII mixtures in the absence of the magnetic field, M_{He} is the magnetic moment of ^{3}He, and $\vec{B}$ is the applied magnetic field. The effective chemical potentials $\mu_\uparrow$ and $\mu_\downarrow$ are determined from the conditions

$$\rho_{3\uparrow} = \frac{1}{\Omega} \sum_{\vec{k}} \langle \hat{n}_{\vec{k}\uparrow} \rangle; \tag{13}$$

and

$$\rho_{3\downarrow} = \frac{1}{\Omega} \sum_{\vec{k}} \langle \hat{n}_{\vec{k}\downarrow} \rangle, \tag{14}$$

$\rho_{3\uparrow}$ ($\rho_{3\downarrow}$) being the number density of spin-up (-down) ^{3}He particles in the mixture.

By summing over the spin indices λ and λ_1, we obtain

$$\begin{aligned}\hat{H} = {} & \sum_{\vec{k}} (\varepsilon(\vec{k}) - \mu_\uparrow) \hat{a}^+_{\vec{k}\uparrow} \hat{a}_{\vec{k}\uparrow} + \sum_{\vec{k}} (\varepsilon(\vec{k}) - \mu_\downarrow) \hat{a}^+_{\vec{k}\downarrow} \hat{a}_{\vec{k}\downarrow} \\ & + \frac{1}{2\Omega} \sum_{\vec{k}\vec{p}\vec{q}} V(q) (\hat{a}^+_{\vec{k}+\vec{q},\downarrow} \hat{a}_{\vec{k}\downarrow} \hat{a}^+_{\vec{p}-\vec{q},\uparrow} \hat{a}_{\vec{p}\uparrow} + \hat{a}^+_{\vec{p}-\vec{q},\uparrow} \hat{a}_{\vec{p}\uparrow} \hat{a}^+_{\vec{k}+\vec{q},\downarrow} \hat{a}_{\vec{k}\downarrow} \\ & + \hat{a}^+_{\vec{k}+\vec{q},\uparrow} \hat{a}^+_{\vec{p}-\vec{q},\uparrow} \hat{a}_{p\uparrow} \hat{a}_{\vec{k}\uparrow} + \hat{a}^+_{\vec{k}+\vec{q},\downarrow} \hat{a}^+_{\vec{p}-\vec{q},\downarrow} \hat{a}_{\vec{p}\downarrow} \hat{a}_{\vec{k}\downarrow}).\end{aligned} \tag{15}$$

The first two terms are the kinetic energies of the spin-up and spin-down particles, respectively. The third and the fourth terms represent the spin-up – spin-down interaction, while the fifth and the sixth terms represent spin-up – spin-up and spin-down – spin-down interactions, respectively. So, we can define the following:

$$\hat{H}_1 = \sum_{\vec{k}} (\varepsilon(\vec{k}) - \mu_\uparrow) \hat{n}_{\vec{k}\uparrow}; \tag{16}$$

$$\hat{H}_2 = \sum_{\vec{k}} (\varepsilon(\vec{k}) - \mu_\downarrow) \hat{n}_{\vec{k}\downarrow}; \tag{17}$$

$$\hat{H}_3 = \frac{1}{2\Omega} \sum_{\vec{k}\vec{p}\vec{q}} V(q) (\hat{a}^+_{\vec{k}+\vec{q},\downarrow} \hat{a}_{\vec{k}\downarrow} \hat{a}^+_{\vec{p}-\vec{q},\uparrow} \hat{a}_{\vec{p}\uparrow}); \tag{18}$$

$$\hat{H}_4 = \frac{1}{2\Omega} \sum_{\vec{k}\vec{p}\vec{q}} V(q) (\hat{a}^+_{\vec{p}-\vec{q},\uparrow} \hat{a}_{\vec{p}\uparrow} \hat{a}^+_{\vec{k}+\vec{q},\downarrow} \hat{a}_{\vec{k}\downarrow}); \tag{19}$$

$$\hat{H}_5 = \frac{1}{2\Omega}\sum_{\vec{k}\vec{p}\vec{q}} V(q)(\hat{a}^{+}_{\vec{k}+\vec{q},\uparrow}\hat{a}^{+}_{\vec{p}-\vec{q},\uparrow}\hat{a}_{p\uparrow}\hat{a}_{\vec{k}\uparrow}); \tag{20}$$

$$\hat{H}_6 = \frac{1}{2\Omega}\sum_{\vec{k}\vec{p}\vec{q}} V(q)(\hat{a}^{+}_{\vec{k}+\vec{q},\downarrow}\hat{a}^{+}_{\vec{p}-\vec{q},\downarrow}\hat{a}_{\vec{p}\downarrow}\hat{a}_{\vec{k}\downarrow}). \tag{21}$$

We note that $\hat{H}_3 = \hat{H}_4$.

Based on the SFA, the Hamiltonian can be expressed as a linear combination of local-field operators $A(k_\downarrow)$, $B(k_\uparrow)$, $C(k_\uparrow)$, $D(k_\downarrow)$, and the number-of-particles operators:

$$\begin{aligned}\hat{H} &= \sum_{k}(\varepsilon(\vec{k}) - \mu_\uparrow)\hat{n}_{\vec{k}\uparrow} + \sum_{\vec{k}}(\varepsilon(\vec{k}) - \mu_\downarrow)\hat{n}_{\vec{k}\downarrow}\\ &\quad + \sum_{k}\hat{A}(\vec{k}_\downarrow)\hat{n}_{\vec{k}\downarrow} + \sum_{k}\hat{C}(\vec{k}_\uparrow)\hat{n}_{\hat{k}\downarrow} + \sum_{\vec{k}}\hat{D}(\vec{k}_\downarrow)\hat{n}_{\vec{k}\downarrow},\end{aligned} \tag{22}$$

where

$$\hat{H}_3 = \sum_{\vec{k}} A(\vec{k}_\downarrow)\hat{n}_{\vec{k}\uparrow}; \tag{23}$$

$$\hat{H}_4 = \sum_{\vec{k}} B(\vec{k}_\uparrow)\hat{n}_{\vec{k}\downarrow}; \tag{24}$$

$$\hat{H}_5 = \sum_{\vec{k}} C(\vec{k}_\uparrow)\hat{n}_{\vec{k}\uparrow}; \tag{25}$$

$$\hat{H}_6 = \sum_{\vec{k}} D(\vec{k}_\downarrow)\hat{n}_{\vec{k}\downarrow}. \tag{26}$$

The local-field operators are assumed to be hermitian and to commute with creation and annihilation operators.

The equation of motion for the spin-up creation operator is

$$\begin{aligned}\frac{d\hat{a}^{+}_{k\uparrow}(\tau)}{d\tau} &= [\hat{H}, \hat{a}^{+}_{k\uparrow}(\tau)] = [\hat{H}_1, \hat{a}^{+}_{k\uparrow}(\tau)] + [\hat{H}_2, \hat{a}^{+}_{k\uparrow}(\tau)] + [\hat{H}_3, \hat{a}^{+}_{k\uparrow}(\tau)]\\ &\quad + [\hat{H}_4, \hat{a}^{+}_{k\uparrow}(\tau)] + [\hat{H}_5, \hat{a}^{+}_{k\uparrow}(\tau)] + [\hat{H}_6, \hat{a}^{+}_{k\uparrow}(\tau)].\end{aligned} \tag{27}$$

We find that $[\hat{H}_2, \hat{a}^{+}_{k\uparrow}(\tau)] = 0$ and $[\hat{H}_6, \hat{a}^{+}_{k\uparrow}(\tau)] = 0$. With $\hat{H}_3 = \hat{H}_4$, we can rewrite Eq. (27) as

$$\frac{d\hat{a}^{+}_{k\uparrow}(\tau)}{d\tau} = [\hat{H}_1, \hat{a}^{+}_{k\uparrow}(\tau)] + [\hat{H}_3, \hat{a}^{+}_{k\uparrow}(\tau)] + [\hat{H}_5, \hat{a}^{+}_{k\uparrow}(\tau)]$$

$$\begin{aligned}
&= (\varepsilon(k) - \mu_\uparrow) a_{k\uparrow}^{+} + \sum_{\vec{q}} [A(\vec{q}_\downarrow)\hat{n}_{q\uparrow}, \hat{a}_{\vec{k}\uparrow}^{+}] + \sum_{\vec{q}} [C(\vec{q}_\uparrow)\hat{n}_{q\uparrow}, \hat{a}_{\vec{k}\uparrow}^{+}] \\
&= (\varepsilon(k) - \mu_\uparrow) \hat{a}_{\vec{k}\uparrow}^{+} + \sum_{\vec{q}} A(\vec{q}_\downarrow)[\hat{n}_{q\uparrow}, \hat{a}_{\vec{k}\uparrow}^{+}] + C(\vec{q}_\uparrow)[\hat{n}_{q\uparrow}, \hat{a}_{\vec{k}\uparrow}^{+}] \\
&= [\varepsilon(k) - \mu_\uparrow + 2\hat{A}(k_\downarrow) + C(k_\uparrow)]\hat{a}_{k\uparrow}^{+}; \\
&= \hat{F}(k_{\uparrow\downarrow})\hat{a}_{k\uparrow}^{+}(\tau),
\end{aligned} \tag{28}$$

where

$$\hat{F}(k_{\uparrow\downarrow}) \equiv \varepsilon(k) - \mu_\uparrow + 2\hat{A}(k_\downarrow) + \hat{C}(k_\uparrow). \tag{29}$$

The local-field operators $\hat{A}(k_\downarrow)$ and $\hat{C}(k_\uparrow)$ are calculated as

$$\begin{aligned}
&[\hat{H}_3, \hat{a}_{k\uparrow}^{+}(\tau)] = \hat{A}(\vec{k}_\downarrow)\hat{a}_{\vec{k}\uparrow}^{+}; \\
&\hat{A}(\vec{k}_\downarrow) = (\hat{a}_{k\uparrow}, [\hat{H}_3, \hat{a}_{k\uparrow}^{+}(\tau)]) = \frac{1}{2\Omega} \sum_{\vec{q}} V(0)\hat{a}_{\vec{q}\downarrow}^{+}\hat{a}_{\vec{q}\downarrow} \\
&= V(0)\frac{N_\downarrow}{2\Omega}.
\end{aligned} \tag{30}$$

where $N_\downarrow$ is the number of the spin-down particles.

In the same way,

$$\hat{C}(\vec{k}_\uparrow) = (\hat{a}_{k\uparrow}, [\hat{H}_5, \hat{a}_{k\uparrow}^{+}(\tau)]) = \frac{1}{\Omega} \sum_{\vec{q}} (V(0) - V(\vec{k} - \vec{q}))\hat{a}_{\vec{q}\uparrow}^{+}\hat{a}_{\vec{q}\uparrow}. \tag{31}$$

Similarly, the equation of motion for the spin-down creation operator is

$$\frac{d\hat{a}_{k\downarrow}^{+}(\tau)}{d\tau} = \hat{G}(\vec{k}_{\uparrow\downarrow})\hat{a}_{k\downarrow}^{+}(\tau), \tag{32}$$

where

$$\hat{G}(k_{\uparrow\downarrow}) \equiv \varepsilon(k) - \mu_\downarrow + 2\hat{B}(k_\uparrow) + \hat{D}(k_\downarrow). \tag{33}$$

We can find the local-field operators $\hat{B}(k_\uparrow)$ and $\hat{D}(k_\downarrow)$ as

$$\hat{B}(\vec{k}_\uparrow) = (\hat{a}_{k\downarrow}, [\hat{H}_4, \hat{a}_{k\downarrow}^{+}(\tau)]) = \frac{1}{2\Omega} \sum_{\vec{q}} V(0)\hat{a}_{\vec{q}\uparrow}^{+}\hat{a}_{\vec{q}\uparrow} = V(0)\frac{N_\uparrow}{2\Omega}; \tag{34}$$

where $N_\uparrow$ is number of the spin-up particles.

$$\hat{D}(\vec{k}_\downarrow) = (\hat{a}_{k\downarrow}, [\hat{H}_6, \hat{a}_{k\downarrow}^{+}(\tau)]) = \frac{1}{\Omega} \sum_{\vec{q}} (V(0) - V(\vec{k} - \vec{q}))\hat{a}_{\vec{q}\downarrow}^{+}\hat{a}_{\vec{q}\downarrow}. \tag{35}$$

The operators $A(k_\downarrow)$ and $B(k_\uparrow)$ are constant field operators; they constitute the 'ether background,' these terms shift the chemical potential for spin-up and spin-down states. The excited local-field operators are

$$\hat{F}_{\text{ex}}(\vec{k}_\uparrow) = \varepsilon(k) - \mu_\uparrow + \frac{1}{\Omega}\sum_{\vec{q}}(V(\vec{q}) - V(\vec{k}-\vec{q}))\hat{a}^{+}_{\vec{q}\uparrow}\hat{a}_{\vec{q}\uparrow}; \tag{36}$$

$$\hat{G}_{\text{ex}}(k_\downarrow) = \varepsilon(k) - \mu_\downarrow + \frac{1}{\Omega}\sum_{\vec{q}}(V(\vec{q}) - V(\vec{k}-\vec{q}))\hat{a}^{+}_{\vec{q}\downarrow}\hat{a}_{\vec{q}\downarrow}. \tag{37}$$

Thus, the excitation grand Hamiltonian describing a neutral spin-polarized fermionic system is

$$\hat{H}_{\text{ex}} = \hat{H} - \hat{H}_0 = \sum_{\vec{k}\uparrow}\hat{F}_{\text{ex}}(\vec{k}_\uparrow)\hat{n}_{\vec{k}\uparrow} + \sum_{\vec{k}\downarrow}\hat{G}_{\text{ex}}(\vec{k}_\downarrow)\hat{n}_{\vec{k}\downarrow}. \tag{38}$$

The general solution of Eq. (28) is

$$\hat{a}^{+}_{\vec{k}\uparrow}(\tau) = \hat{a}^{+}_{\vec{k}\uparrow}\exp[\hat{F}_{\text{ex}}(\vec{k}_\uparrow)\tau]; \tag{39}$$

$$\hat{F}_{\text{ex}}(k_\uparrow) = \langle\hat{F}_{\text{ex}}(k_\uparrow)\rangle + \Delta\hat{F}_{\text{ex}}(k_\uparrow).$$

Substituting this into Eq. (39), we obtain

$$\begin{aligned}\hat{a}^{+}_{\vec{k}\uparrow}(\tau) &= \hat{a}^{+}_{\vec{k}\uparrow}\exp[(\langle\hat{F}_{\text{ex}}(k_\uparrow)\rangle + \Delta F_{\text{ex}}(k_\uparrow))t] \\ &= \hat{a}^{+}_{\vec{k}\uparrow}\exp[\langle\hat{F}_{\text{ex}}(k_\uparrow)\rangle\tau]\exp(\Delta\hat{F}_{\text{ex}}(k_\uparrow)\tau).\end{aligned} \tag{40}$$

In the SFA, the square of the quadratic fluctuation operator can be replaced with its mean value:

$$(\Delta\hat{F}_{\text{ex}}(k_\uparrow))^2 = \langle(\Delta\hat{F}_{\text{ex}}(k_\uparrow))^2\rangle = \varphi^2_{\text{F}}((k_\uparrow)); \tag{41}$$

$$(\Delta\hat{G}_{\text{ex}}(k_\downarrow))^2 = \langle(\hat{G}_{\text{ex}}(k_\downarrow))^2\rangle = \varphi^2_{G}((k_\downarrow)). \tag{42}$$

It is more convenient to rewrite (40) as linear in terms of the fluctuations of the local-field operator; this can be done with the aid of the identity:

$$B(a + b\Delta\hat{F}_{\text{ex}}(k_\uparrow)) \equiv \eta_{0\uparrow}(k) + \eta_{1\uparrow}(k)\Delta\hat{F}_{\text{ex}}(k_\uparrow), \tag{43}$$

where

$$\eta_{0\uparrow}(k) = \frac{1}{2}[B(a + b\varphi_{\text{F}}(k_\uparrow) + [B(a - b\varphi_{\text{F}}(k_\uparrow)]; \tag{44}$$

$$\eta_{1\uparrow}(k) = \frac{1}{2\varphi_{\text{F}}}[B(a + b\varphi_{\text{F}}(k_\uparrow) - [B(a - b\varphi_{\text{F}}(k_\uparrow)]. \tag{45}$$

According to this identity, we can write (40) in the final form:

$$\hat{a}^+_{\vec{k}\uparrow}(\tau) = \hat{a}^+_{\vec{k}\uparrow}\exp(\langle\hat{F}_{\text{ex}}(k_\uparrow)\rangle\tau)\left[\cosh(\varphi_{\text{F}}(k_\uparrow)\tau) + \frac{\Delta\hat{F}_{\text{ex}}(k_\uparrow)}{\varphi_{\text{F}}(k_\uparrow)}\sinh(\varphi_{\text{F}}(k_\uparrow)\tau)\right]. \tag{46}$$

By using the identity

$$\langle\hat{C}(\beta)\hat{B}\rangle \equiv \langle\hat{B}\hat{C}\rangle \equiv \frac{1}{Q}\text{Tr}[\exp(-\beta\hat{H})\hat{B}\hat{C}], \tag{47}$$

Q being grand partition function of the system; then replacing τ with $\beta \equiv 1/k_{\text{B}}T$, k_{B} being Boltzmann's constant and T the absolute temperature; and invoking Eq. (46), we can obtain the so-called long-range equation (Al-Sugheir and Ghassib, 2002):

$$\langle\hat{n}_{k\uparrow}\hat{A}\rangle = \eta_{0\uparrow}(k)\langle\hat{A}\rangle + \eta_{1\uparrow}(k)\langle\Delta\hat{F}_{\text{ex}}(k_\uparrow)\hat{A}\rangle. \tag{48}$$

The operator $\hat{A}$ is chosen to commute with the creation and annihilation operators as well as local-field operators:

$$\eta_{0\uparrow}(k) = \frac{1}{2}\left\{\frac{1}{\exp[\beta(\langle\hat{F}_{\text{ex}}(k_\uparrow)\rangle + \varphi_{\text{F}}(k_\uparrow))] + 1} + \frac{1}{\exp[\beta(\langle\hat{F}_{\text{ex}}(k_\uparrow)\rangle - \varphi_{\text{F}}(k_\uparrow))] + 1}\right\}; \tag{49}$$

and

$$\eta_{1\uparrow}(k) = \frac{1}{2\varphi_{\text{F}}(k_\uparrow)}\left\{\frac{1}{\exp[\beta(\langle\hat{F}_{\text{ex}}(k_\uparrow)\rangle + \varphi_{\text{F}}(k_\uparrow))] + 1} - \frac{1}{\exp[\beta(\langle\hat{F}_{\text{ex}}(k_\uparrow)\rangle - \varphi_{\text{F}}(k_\uparrow))] + 1}\right\}. \tag{50}$$

Similarly, we can obtain the following equations for the spin-down subsystem:

$$\hat{a}^+_{\vec{k}\downarrow}(\tau) = \hat{a}^+_{\vec{k}\downarrow}\exp(\langle\hat{G}_{\text{ex}}(k_\downarrow)\rangle\tau)\left[\cosh(\varphi_G(k_\downarrow)\tau) + \frac{\Delta\hat{G}_{\text{ex}}(k_\downarrow)}{\varphi_G(k_\downarrow)}\sinh(\varphi_G(k_\downarrow)\tau)\right]; \tag{51}$$

$$\langle\hat{n}_{k\downarrow}\hat{A}\rangle = \eta_{0\downarrow}(k)\langle\hat{A}\rangle + \eta_{1\downarrow}(k)\langle\Delta\hat{G}_{\text{ex}}(k_\downarrow)\hat{A}\rangle, \tag{52}$$

where

$$\eta_{0\downarrow}(k) = \frac{1}{2}\left\{\frac{1}{\exp[\beta(\langle\hat{G}_{\text{ex}}(k_\downarrow)\rangle + \varphi_G(k_\downarrow))] + 1} + \frac{1}{\exp[\beta(\langle\hat{G}_{\text{ex}}(k_\downarrow)\rangle - \varphi_G(k_\downarrow))] + 1}\right\}; \tag{53}$$

and

$$\eta_{1\downarrow}(k) = \frac{1}{2\varphi_G(k_\downarrow)}\left\{\frac{1}{\exp[\beta(\langle\hat{G}_{\text{ex}}(k_\downarrow)\rangle + \varphi_G(k_\downarrow))] + 1} - \frac{1}{\exp[\beta(\langle\hat{G}_{\text{ex}}(k_\downarrow)\rangle - \varphi_G(k_\downarrow))] + 1}\right\}. \tag{54}$$

We can now find the closed system of nonlinear integral equations for spin-up and spin-down subsystems from the long-range equations. First, putting $\hat{A} = 1$ in Eq. (48), we have the spin-up particle distribution, where the quadratic fluctuations are symmetric, $\langle\Delta\hat{F}_{\text{ex}}(k_\uparrow)\rangle = 0$:

$$\langle\hat{n}_{k\uparrow}\rangle = \eta_{0\uparrow}(k). \tag{55}$$

Substituting Eq. (55) in the long-range equation (48), we obtain

$$\langle\Delta\hat{n}_{k\uparrow}\hat{A}\rangle = -\eta_{1\uparrow}(k)\langle\Delta\hat{F}_{\text{ex}}(k_\uparrow)\hat{A}\rangle. \tag{56}$$

Putting $\hat{A} = \Delta\hat{n}_{q\lambda}$ in this equation, where $q \neq k$, we have the pair correlation function $\langle\Delta\hat{n}_{k\uparrow}\Delta\hat{n}_{q\lambda}\rangle_c$, the index c denoting the true correlations $q \neq k$ or $\lambda \neq \uparrow$:

$$\begin{aligned}\langle\Delta\hat{n}_{k\uparrow}\Delta\hat{n}_{q\lambda}\rangle_c &= -\eta_{1\uparrow}(k)\langle\Delta\hat{F}_{\text{ex}}(k_\uparrow)\Delta\hat{n}_{q\lambda}\rangle_c \\ &= -\frac{\eta_{1\uparrow}(k)}{\Omega}\sum_{\vec{p}} W(\vec{k},\vec{p})\langle\Delta\hat{n}_{p\uparrow}\Delta\hat{n}_{q\lambda}\rangle,\end{aligned} \tag{57}$$

where

$$W(\vec{k},\vec{p}) \equiv V(\vec{p}) - V(\vec{k}-\vec{p}).$$

Now, If $k \neq q$ then $\lambda = \uparrow$ or $\lambda = \downarrow$; and we find the pair correlation function:

$$\langle\Delta\hat{n}_{k\uparrow}\Delta\hat{n}_{q\uparrow}\rangle_c = -\frac{\eta_{1\uparrow}(k)}{\Omega}\sum_{\vec{p}} W(\vec{k},\vec{p})\langle\Delta\hat{n}_{p\uparrow}\Delta\hat{n}_{q\uparrow}\rangle; \tag{58}$$

$$\langle\Delta\hat{n}_{k\uparrow}\Delta\hat{n}_{q\downarrow}\rangle_c = -\frac{\eta_{1\uparrow}(k)}{\Omega}\sum_{\vec{p}} W(\vec{k},\vec{p})\langle\Delta\hat{n}_{p\uparrow}\Delta\hat{n}_{q\downarrow}\rangle. \tag{59}$$

The closed set of nonlinear integral equations for the spin-polarized ^{3}He–HeII mixture is, then:

For the spin-up subsystem:

$$\hat{F}_{\text{ex}}(\vec{k}_\uparrow) = \varepsilon(k) - \mu_\uparrow + \frac{1}{\Omega}\sum_{\vec{q}} W(\vec{k},\vec{q})\hat{a}^+_{\vec{q}\uparrow}\hat{a}_{\vec{q}\uparrow}; \tag{60}$$

$$\langle n_{k\uparrow}\rangle = \eta_{0\uparrow}(k) \tag{61}$$

$$
\begin{aligned}
\langle(\Delta n_{k\uparrow})^2\rangle &= -k_\beta T\frac{\partial\langle n_{k\uparrow}\rangle}{\partial\mu_\uparrow} \\
&= \langle n_{k\uparrow}\rangle\left(1-\langle n_{k\uparrow}\rangle\right); \qquad (62)
\end{aligned}
$$

$$\langle\Delta n_{k\uparrow}\Delta n_{q\uparrow}\rangle_c = -\frac{\eta_{1\uparrow}(k)}{\Omega}\sum_{\vec{p}} W(\vec{k},\vec{p})\langle\Delta n_{p\uparrow}\Delta n_{q\uparrow}\rangle; \qquad (63)$$

$$\langle\Delta n_{p\uparrow}\Delta n_{q\uparrow}\rangle = \langle(\Delta n_{q\uparrow})^2\rangle\delta_{pq} + \langle\Delta n_{p\uparrow}\Delta n_{q\uparrow}\rangle_c; \qquad (64)$$

and substituting $\hat{A} = \Delta\hat{F}_{\text{ex}}$ in Eq. (56), we have

$$\eta_{1\uparrow}(k)\varphi_{\text{F}}^2(k_\uparrow) = -\frac{1}{\Omega}\sum_{\vec{p}} W(\vec{k},\vec{p})\langle\Delta n_{k\uparrow}\Delta n_{p\uparrow}\rangle_c. \qquad (65)$$

In a similar manner, the closed set of nonlinear integral equations for the spin-down subsystem is:

$$\hat{G}_{\text{ex}}(k_\downarrow) = \varepsilon(k) - \mu_\downarrow + \frac{1}{\Omega}\sum_{\vec{q}} W(\vec{k},\vec{q})\hat{a}^+_{\vec{q}\downarrow}\hat{a}_{\vec{q}\downarrow}; \qquad (66)$$

$$
\begin{aligned}
\langle n_{k\downarrow}\rangle &= \eta_{0\downarrow}(k) \qquad (67) \\
\langle(\Delta n_{k\downarrow})^2\rangle &= -k_\beta T\frac{\partial\langle n_{k\downarrow}\rangle}{\partial\mu_\downarrow} \\
&= \langle n_{k\downarrow}\rangle\left(1-\langle n_{k\downarrow}\rangle\right);
\end{aligned}
$$

$$\langle\Delta n_{k\downarrow}\Delta n_{q\downarrow}\rangle_c = -\frac{\eta_{1\downarrow}(k)}{\Omega}\sum_{\vec{p}} W(\vec{k},\vec{p})\langle\Delta n_{p\downarrow}\Delta n_{q\downarrow}\rangle; \qquad (68)$$

$$\langle\Delta n_{p\downarrow}\Delta n_{q\downarrow}\rangle = \langle(\Delta n_{q\downarrow})^2\rangle\delta_{pq} + \langle\Delta n_{p\downarrow}\Delta n_{q\downarrow}\rangle_c; \qquad (69)$$

and

$$\eta_{1\downarrow}(k)\varphi_G^2(k_\downarrow) = -\frac{1}{\Omega}\sum_{\vec{p}} W(\vec{k},\vec{p})\langle\Delta n_{k\downarrow}\Delta n_{p\downarrow}\rangle_c. \qquad (70)$$

We now have the closed system of nonlinear integral equations consisting of $\langle\hat{F}_{\text{ex}}(k_\uparrow)\rangle$, $\langle\hat{n}_{k\uparrow}\rangle$, $\langle(\Delta\hat{n}_{k\uparrow})^2\rangle$, $\langle\Delta\hat{n}_{k\uparrow}\Delta\hat{n}_{q\uparrow}\rangle_c$, $\varphi_{\text{F}}(k_\downarrow)$ for the spin-up system; and $\langle\hat{G}_{\text{ex}}(k_\downarrow)\rangle$, $\langle\hat{n}_{k\downarrow}\rangle$, $\langle(\Delta\hat{n}_{k\downarrow})^2\rangle$, $\langle\Delta\hat{n}_{k\downarrow}\Delta\hat{n}_{q\downarrow}\rangle_c$, $\varphi_G(k_\downarrow)$ for the spin-down system. These nonlinear integral equations will be solved numerically by Gaussian quadrature (Ali, 1997; Bishop *et al.*, 1977; Burden and Faires, 1993; Ghassib *et al.*, 1976).

To calculate the thermodynamic properties of the system, we should first calculate the grand partition function Q. The usual definition is

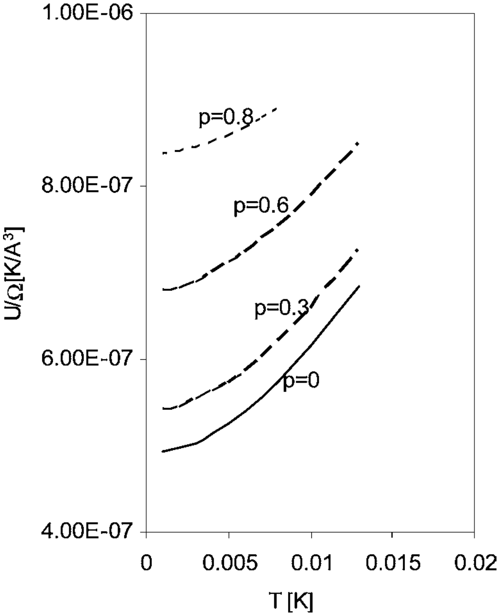

Fig. 1. The mean internal energy per unit volume U/Ω for the spin-polarized ^{3}He–HeII mixture as a function of temperature T at different values of spin polarization p.

$$Q = \mathrm{Tr}(\exp(-\beta\hat{H})$$

$$= \sum_{n_{k\uparrow},n_{k\downarrow}} \exp\left[-\beta(\sum_{\vec{k}} \hat{F}_{\mathrm{ex}}(k_\uparrow)\hat{n}_{k\uparrow} + \sum_{\vec{k}}(\hat{G}_{\mathrm{ex}}(k_\downarrow)\hat{n}_{k\downarrow})\right] \tag{71}$$

$$= \prod_{\vec{k}} \sum_{n_{k\uparrow},n_{k\downarrow}} \exp(-\beta(\hat{F}_{\mathrm{ex}}(k_\uparrow)\hat{n}_{k\uparrow} + \hat{G}_{\mathrm{ex}}(k_\downarrow)\hat{n}_{k\downarrow})). \tag{72}$$

For Fermi systems, $n_{k\uparrow} = 0, 1$; $n_{k\downarrow} = 0, 1$; so that Eq. (72) becomes

$$Q = \prod_{\vec{k}} [1 + \exp(-\beta\hat{F}_{\mathrm{ex}}(k_\uparrow))][1 + \exp(-\beta\hat{G}_{\mathrm{ex}}(k_\downarrow))]$$

$$= \prod_{\vec{k}} [1 + \exp(-\beta\hat{F}_{\mathrm{ex}}(k_\uparrow))] \prod_{\vec{k}} [1 + \exp(-\beta\hat{G}_{\mathrm{ex}}(k_\downarrow))]. \tag{73}$$

The grand partition function Q is the product of the grand partition functions of the two subsystems:

$$Q = Q_\uparrow Q_\downarrow,$$

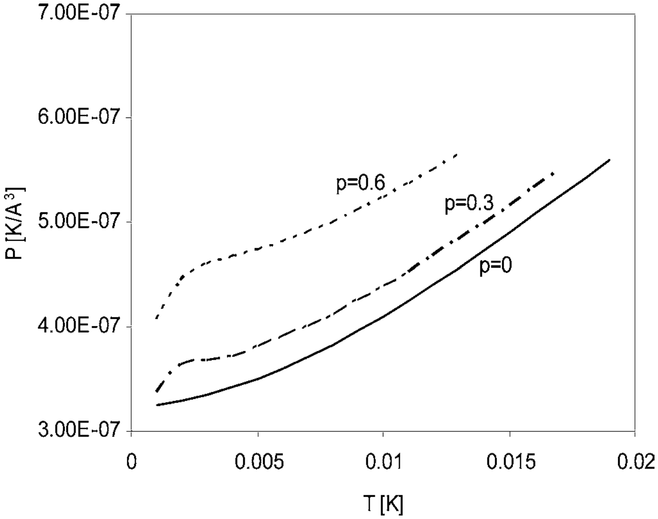

Fig. 2. The pressure P for the spin-polarized ^{3}He–HeII mixture as a function of temperature T at different values of spin polarization p.

where

$$Q_\uparrow = \prod_{\vec{k}} [1 + \exp(-\beta \hat{F}_{\text{ex}}(k_\uparrow))]; \tag{74}$$

and

$$Q_\downarrow = \prod_{\vec{k}} [1 + \exp(-\beta \hat{G}_{\text{ex}}(k_\downarrow))]. \tag{75}$$

It is more convenient to take the logarithms of both sides:

$$\begin{aligned}\ln Q_\uparrow &= \ln \prod_{\vec{k}} [1 + \exp(-\beta \hat{F}_{\text{ex}}(k_\uparrow))] \\ &= \sum_{\vec{k}} \ln[1 + \exp(-\beta \hat{F}_{\text{ex}}(k_\uparrow))] \\ &= \sum_{\vec{k}} \langle q_{0\uparrow}(k) + q_{1\uparrow}(k)\Delta \hat{F}_{\text{ex}}(k_\uparrow)\rangle\end{aligned}$$

From the symmetry of the quadratic fluctuations of the local-field operator, we get

$$\ln Q_\uparrow = \sum_{\vec{k}} q_{0\uparrow}(k); \tag{76}$$

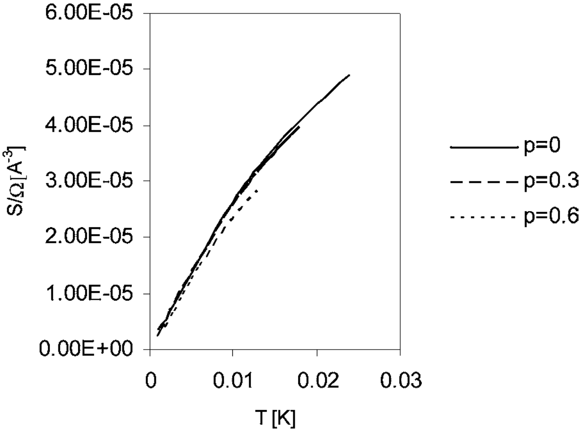

Fig. 3. The entropy per unit volume S/Ω for the spin-polarized ^{3}He–HeII mixture as a function of temperature T at different values of spin polarization p.

where

$$q_{0\uparrow}(k) = \frac{1}{2}\ln[(1+\exp(-\beta(\langle \hat{F}_{\text{ex}}(k_\uparrow)\rangle + \varphi_{\text{F}}(k_\uparrow)))) \times (1+\exp(-\beta(\langle \hat{F}_{\text{ex}}(k_\uparrow)\rangle - \varphi_{\text{F}}(k_\uparrow))))]. \tag{77}$$

Similarly, for the spin-down subsystem:

$$\ln Q_\downarrow = \sum_{\vec{k}} q_{0\downarrow}(k),$$

where

$$q_{0\downarrow}(k) = \frac{1}{2}\ln[(1+\exp(-\beta(\langle \hat{G}_{\text{ex}}(k_\downarrow)\rangle + \varphi_{G}(k_\downarrow)))) \times (1+\exp(-\beta(\langle G_{\text{ex}}(k_\downarrow)\rangle - \varphi_{G}(k_\downarrow))))]. \tag{78}$$

The logarithm of the total grand partition function is, then,

$$\ln Q = \ln Q_\uparrow + \ln Q_\downarrow = \sum_{\vec{k}} q_{0\uparrow} + \sum_{\vec{k}} q_{0\downarrow}(k). \tag{79}$$

The mean internal energy is

$$U = -\left(\frac{\partial \ln Q}{\partial \beta}\right)_{Z,\Omega}, \tag{80}$$

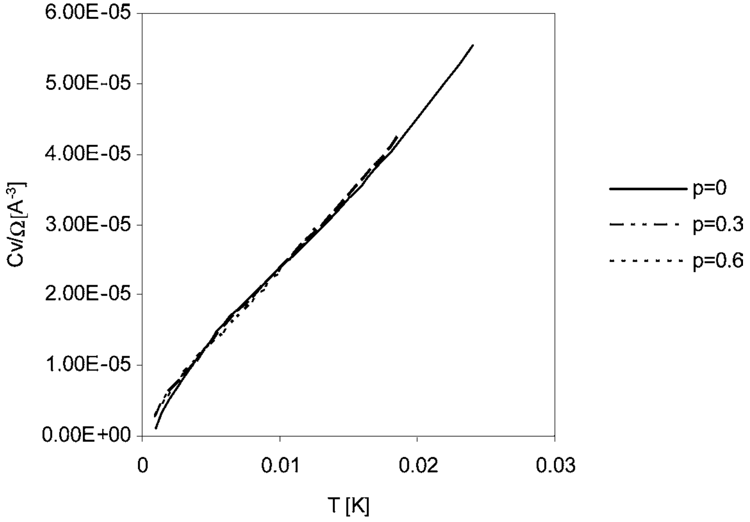

Fig. 4. The specific heat capacity per unit volume C_v/Ω for the spin-polarized ^{3}He–HeII mixture as a function of temperature T at different values of spin polarization p.

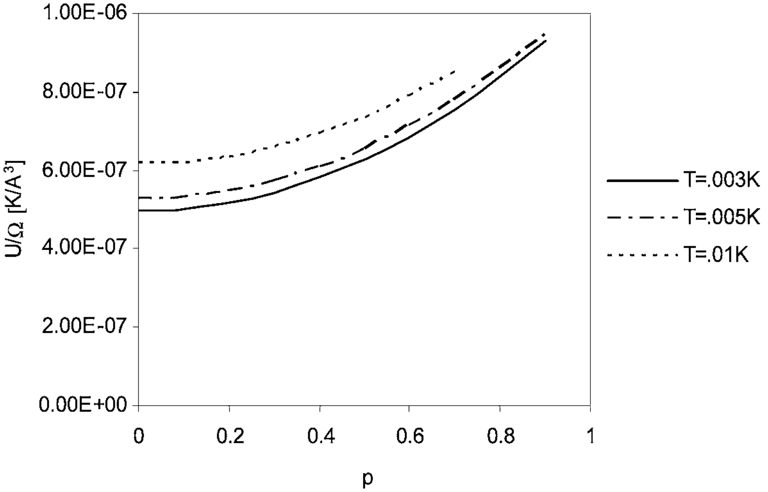

Fig. 5. The mean internal energy per unit volume U/Ω for the spin-polarized ^{3}He–HeII mixture as a function of spin polarization p at different values of temperature T.

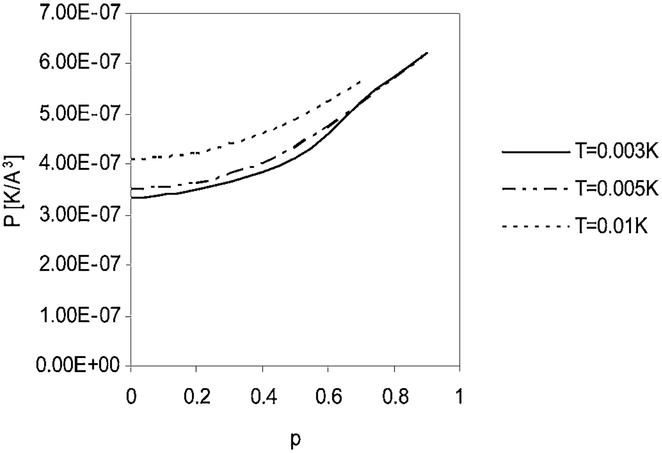

Fig. 6. The pressure P for the spin-polarized ^{3}He–HeII mixture as a function of spin polarization p at different values of temperature T.

where $z \equiv \exp(\beta\mu)$ is the fugacity;

$$\left(\frac{\partial q_{0\uparrow}(k)}{\partial\beta}\right)_{Z_\downarrow,\Omega} = \frac{1}{2}\left(\frac{\langle\hat{F}_{\text{ex}}(k_\uparrow)\rangle + \mu_\uparrow + \varphi_{\text{F}}(k_\uparrow)}{\exp(\beta(\langle\hat{F}_{\text{ex}}(k_\uparrow)\rangle + \varphi_{\text{F}}(k_\uparrow))) + 1} + \frac{\langle\hat{F}_{\text{ex}}(k_\uparrow)\rangle + \mu_\uparrow - \varphi_{\text{F}}(k_\uparrow)}{\exp(\beta(\langle\hat{F}_{\text{ex}}(k_\uparrow)\rangle - \varphi_{\text{F}}(k_\uparrow))) + 1}\right); \tag{81}$$

$$\left(\frac{\partial q_{0\downarrow}(k)}{\partial\beta}\right)_{Z_\downarrow,\Omega} = \frac{1}{2}\left(\frac{\langle\hat{G}_{\text{ex}}(k_\downarrow)\rangle + \mu_\downarrow + \varphi_G(k_\downarrow)}{\exp(\beta(\langle\hat{G}_{\text{ex}}(k_\downarrow)\rangle + \varphi_G(k_\downarrow))) + 1} + \frac{\langle\hat{G}_{\text{ex}}(k_\uparrow)\rangle + \mu_\downarrow - \varphi_G(k_\downarrow)}{\exp(\beta(\langle\hat{G}_{\text{ex}}(k_\uparrow)\rangle - \varphi_G(k_\downarrow))) + 1}\right); \tag{82}$$

$$U = \sum_{\vec{k}}\left(\frac{\partial q_{0\uparrow}(k)}{\partial\beta}\right)_{Z_\uparrow} + \sum_{k}\left(\frac{\partial q_{0\downarrow}(k)}{\partial\beta}\right)_{Z_\downarrow}. \tag{83}$$

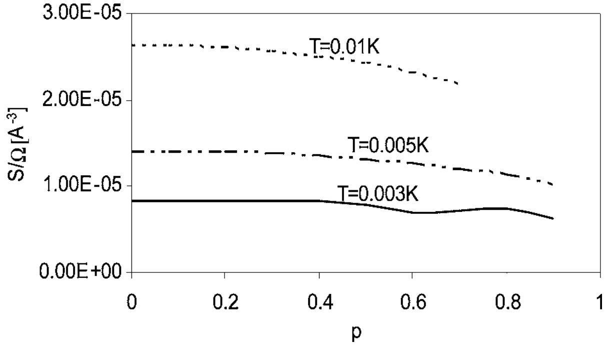

Fig. 7. The entropy per unit volume S/Ω for the spin-polarized ^{3}He–HeII mixture as a function of spin polarization p at different values of temperature T.

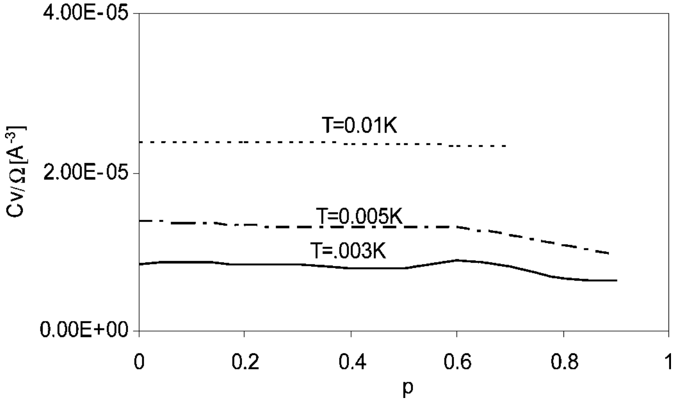

Fig. 8. The specific heat capacity per unit volume C_v/Ω for the spin-polarized ^{3}He–HeII mixture as a function of spin polarization p at different values of temperature T.

From the grand partition function and the mean internal energy, it is simple to evaluate the other thermodynamic properties:

The usual definition of the pressure is given by (Huang, 1987; Pathria, 2004)

$$P = k_B T \frac{\ln Q}{\Omega}. \tag{84}$$

The entropy of the system can be evaluated from the first law of thermodynamics:

$$S = \frac{1}{T}(U + P\Omega - \mu_\uparrow N_\uparrow - \mu_\downarrow N_\downarrow). \tag{85}$$

The specific heat capacity of the system at constant volume is

$$C_v = \left(\frac{\partial \left(U - \mu_\uparrow N_\uparrow - \mu_\downarrow N_\downarrow \right)}{\partial T} \right)_\Omega . \tag{86}$$

Finally, the chemical potentials of the spin-up and spin-down subsystems, which play a crucial role in calculating the functions $\eta_{0\uparrow}(k)$, $\eta_{1\uparrow}(k)$, $\eta_{0\downarrow}(k)$, and $\eta_{1\downarrow}(k)$ are calculated according the conditions

$$\rho_{3\uparrow} = \frac{1}{\Omega} \sum_{\vec{k}} \langle \hat{n}_{\vec{k}\uparrow} \rangle; \qquad \rho_{3\downarrow} = \frac{1}{\Omega} \sum_{\vec{k}} \langle \hat{n}_{\vec{k}\downarrow} \rangle, \tag{87}$$

where $\rho_{3\uparrow}$ ($\rho_{3\downarrow}$) is the number density of spin-up (-down) ^{3}He particles in the mixture.

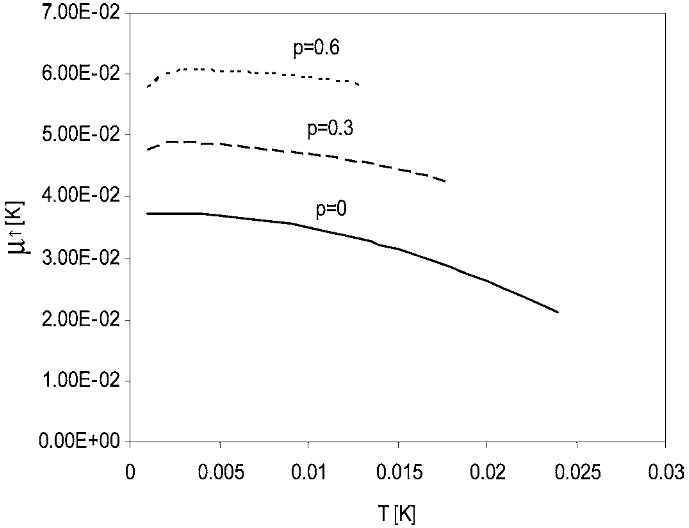

Fig. 9. The chemical potential for the spin-up subsystem of spin-polarized ^{3}He–HeII mixtures as a function of temperature T at different values of spin polarization p.

The number density of ^{3}He particles in HeII for volume differential coefficient $\alpha = 0.284$ is given by

$$\rho_3 = \frac{6.022 \times 10^{29} x}{27.58(1 + 0.284x)} \quad (\text{atoms}\,\text{m}^{-3}), \tag{88}$$

x being the concentration of ^{3}He in HeII.

3. CALCULATIONS

In the thermodynamic limit, the summation in the closed sets for spin-up and spin-down particles can be changed to integration. After integrating over the solid angle, the closed set of equations for the spin-up subsystem becomes

$$\langle \hat{F}_{\text{ex}}(k_\uparrow) \rangle = \varepsilon(k) - \mu_\uparrow + \frac{1}{2\pi^2} \int_0^\infty W(k, p) \langle \hat{n}_{q\uparrow} \rangle p^2 dp; \tag{89}$$

$$\langle \Delta n_{k\uparrow} \Delta n_{q\uparrow} \rangle_c = -\frac{\eta_{1\uparrow}(k)}{2\pi^2} \int_0^\infty W(k, p)) \langle \Delta \hat{n}_{p\uparrow} \Delta \hat{n}_{q\uparrow} \rangle p^2 dp; \tag{90}$$

$$\eta_{1\uparrow}(k) \varphi_{\text{F}}^2(k_\uparrow) = -\frac{1}{2\pi^2} \int_0^\infty W(k, p) \langle \Delta n_{k\uparrow} \Delta n_{p\uparrow} \rangle_c p^2 dp. \tag{91}$$

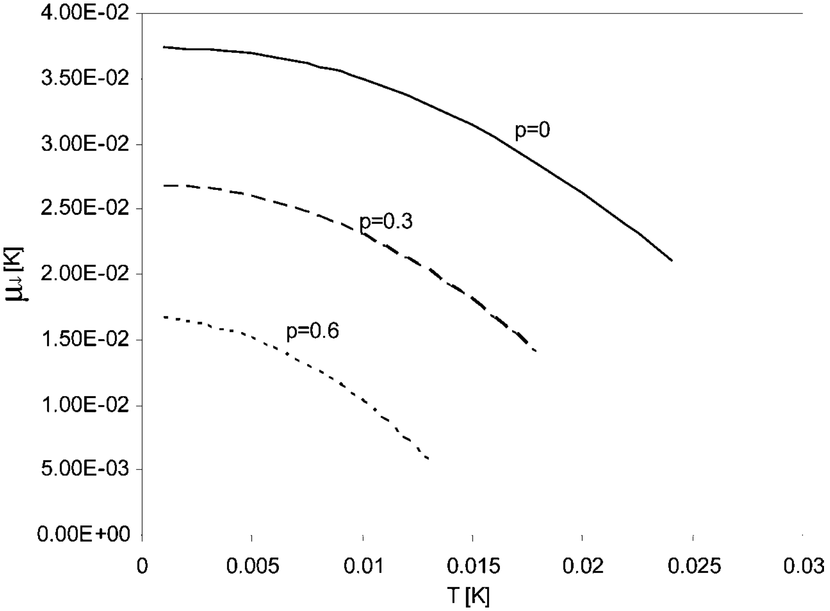

Fig. 10. The chemical potential for the spin-down subsystem of spin-polarized ^{3}He–HeII mixtures as a function of temperature T at different values of spin polarization p.

For the spin-down subsystem:

$$\langle \hat{G}_{\text{ex}}(k_\downarrow)\rangle = \varepsilon(k) - \mu_\downarrow + \frac{1}{2\pi^2}\int_0^\infty W(k,p)\langle \hat{n}_{q\downarrow}\rangle p^2 dp; \tag{92}$$

$$\langle \Delta n_{k\downarrow}\Delta n_{q\downarrow}\rangle_c = -\frac{\eta_{1\downarrow}(k)}{2\pi^2}\int_0^\infty W(k,p)\langle \Delta n_{p\downarrow}\Delta n_{q\downarrow}\rangle p^2 dp; \tag{93}$$

$$\eta_{1\downarrow}(k)\varphi_G^2(k_\downarrow) = -\frac{1}{2\pi^2}\int_0^\infty W(k,p)\langle \Delta n_{k\downarrow}\Delta n_{p\downarrow}\rangle_c p^2 dp, \tag{94}$$

where $W(k,p) \equiv V(p,0) - V(k,p)$, $V(k,p)$ being the Fourier–Bessel transform of the potential, defined as

$$V(k,p) = 4\pi\int_0^\infty V(r)\frac{\sin(kr)}{kr}\frac{\sin(pr)}{pr}r^2 dr. \tag{95}$$

The integrands in (89)–(94) are calculated by Gaussian quadrature (Ali, 1997; Bishop *et al.*, 1977; Ghassib *et al.*, 1976). Our set of nonlinear integral equations have been solved numerically by an iteration method for a realistic interhelium potential. The effective interaction in configuration space between two ^{3}He quasiparticles embedded in HeII is the sum of three physical effects (Campbell, 1967).

Table I. Fermi Temperatures for Spin-Up and Spin-Down Subsystems at Different Values of Spin Polarization p

Polarization, p	$T_{F\uparrow}$ (mK)	$T_{F\downarrow}$ (mK)
0	37.3	37.3
0.2	45.4	29.8
0.4	53.4	24.6
0.6	58.1	16.7

The first effect is the direct ^{3}He–HeII interaction, V_{33}. To this end, we have used one of the most highly acclaimed interatomic helium potentials, the so-called HFDHE2 (Aziz *et al.*, 1979; Janzen and Aziz, 1995):

$$V_{33}(r) = \varepsilon A \exp(-\beta x) - \left(\frac{C_6}{x^6} + \frac{C_8}{x^8} + \frac{C_{10}}{x^{10}}\right) \varepsilon f(x); \tag{96}$$

where

$$f(x) = \begin{Bmatrix} \exp\left[-\left(\frac{D}{x} - 1\right)^2\right], & x < D \\ 1, & x \geq D \end{Bmatrix}; \tag{97}$$

$x \equiv \frac{r}{r_m}$; $x_m = 2.98\,\text{Å}$; $D = 1.24$; $A = 0.554 \times 10^6$; $\beta = 13.35$; $C_6 = 1.373$; $C_8 = 0.425$; $C_{10} = 0.178$; $\varepsilon = 10.8\,\text{K}$.

The second effect is the induced potential arising from the interaction between the ^{3}He atom with the HeII background through the bare potential V_{34}:

$$V_{34}(r) = -2(1+\alpha)V_{34}([r_{34}])g(r), \tag{98}$$

where

$$g(r) = \begin{Bmatrix} \frac{3y}{4} - \frac{y^3}{16}, & y \leq 2 \\ 1, & y > 2 \end{Bmatrix}, \tag{99}$$

with

$$y \equiv r\left(\frac{4\pi}{3\omega_3}\right)^{1/3}; \quad \omega_3 = (1+\alpha)\omega_4. \tag{100}$$

Here, ω_3 (ω_4) is the volume per particle occupied by a ^{3}He (^{4}He) atom, and $[r_{34}] \equiv \max(4.0, r)$. The third effect is associated with the induced potential HeII–HeII background interaction through the bare potential V_{44}:

$$V_{44} = (1+\alpha)^2 V_{44}([r_{44}])g^2(r), \tag{101}$$

where $[r_{44}] \equiv \max(3.8, r)$.

The total effective interatomic potential between two ^{3}He atoms is, therefore,

$$V(r) = V_{33}(r) + V_{34}(r) + V_{44}(r). \tag{102}$$

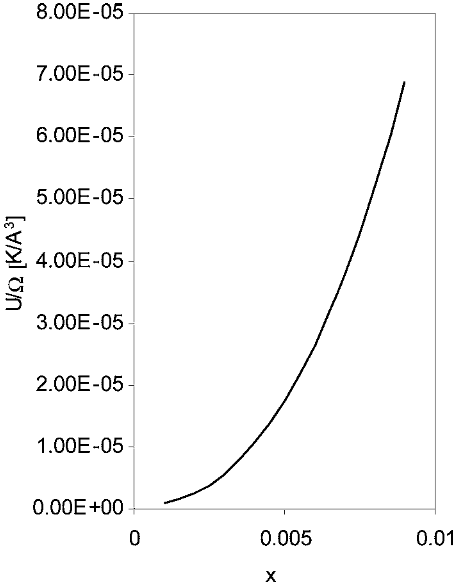

Fig. 11. The mean internal energy per unit volume U/Ω for the spin-polarized ^{3}He–HeII mixture as a function of concentration at temperature $T = 0.02$ K and spin polarization $p = 0.2$.

The Fourier–Bessel transform of the effective interatomic potential is calculated by the same method for the bare potential HFDHE2 (Aziz *et al.*, 1979; Janzen and Aziz, 1995). Throughout our calculations the effective mass $m_3^* = 2.35m_3$ and a natural system of units is used, such that $\hbar = 1 = m_3$ where m_3 is ^{3}He atomic mass, the conversion factor being $\hbar^2/m_3 = 16.0838$ K.Å^2 (Bishop *et al.*, 1977).

4. RESULTS AND DISCUSSION

The thermodynamic properties of the dilute spin-polarized ^{3}He–HeII mixture for a ^{3}He concentration of 0.1% were studied at temperature below the Fermi temperature at different values of temperatures and spin polarization. Also, the mean internal energy per unit volume and pressure were studied as functions of concentration at constant temperature and spin polarization.

The mean internal energy per unit volume, pressure, entropy per unit volume, specific heat per unit volume, and chemical potential were calculated at different temperatures and at different values of spin polarization.

Figures 1–4 show the numerical calculations of the thermodynamic properties of the spin-polarized ^{3}He–HeII mixture as functions of temperature T at different

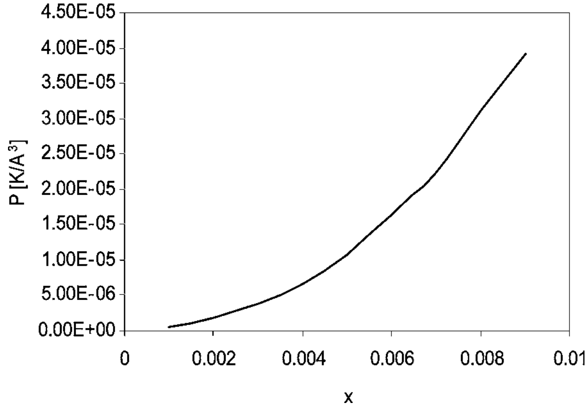

Fig. 12. The pressure P for the spin-polarized ^{3}He–HeII mixture as a function of concentration at temperature $T = 0.02$ K and spin polarization $p = 0.2$.

values of spin polarization. The mean internal energy per unit volume, the pressure, the entropy per unit volume, and the specific heat per unit volume increase with increasing temperature.

Figures 5–8 show the numerical calculations of the thermodynamic properties of spin-polarized ^{3}He–HeII mixtures as functions of spin polarization p at different values of temperature. From these figures, we note that the mean internal energy per unit volume and the pressure increase with increasing spin polarization; while the entropy per unit volume and the specific heat per unit volume are weakly dependent on spin polarization.

Figures 9 and 10 show the effective chemical potentials as functions of temperature for spin-up and spin-down subsystems at different values of spin polarization. The increase in the chemical potential of the spin-up subsystem with polarization is obviously caused by the increase of the density of spin-up particles; while the chemical potential of the spin-down subsystem deceases because of the decrease in density of spin-down particles. From these figures, we note that the two subsystems of spin-polarized mixtures have different Fermi temperatures $T_{\mathrm{F}\uparrow}$ and $T_{F\downarrow}$, $T_{\mathrm{F}\uparrow}$ being larger the $T_{\mathrm{F}\downarrow}$ as shown in Table I. If $T < T_{\mathrm{F}\downarrow}$ the two subsystems are degenerate; but if $T > T_{\mathrm{F}\uparrow}$, the two subsystems are classical. The interesting feature is that when $T_{\mathrm{F}\downarrow} < T < T_{\mathrm{F}\uparrow}$, there is a region where the spin-up subsystem is degenerate and the spin-down subsystem is classical.

Figures 11 and 12 show the numerical calculations of the mean internal energy per unit volume and the pressure of the spin-polarized ^{3}He–HeII mixture as functions of concentration at constant temperature and spin polarization.

5. CONCLUSION

The SFA has been used to study spin-polarized ^{3}He–HeII mixtures for the first time. It has been found that the SFA is valid for this system at low temperatures ($\leq$0.025 K). We have found that the spin-polarized ^{3}He atoms exist in two subsystems: spin-up and spin-down. The basic achievements of this paper are, then, (1) the full derivation of the SFA, for the first time, for spin-polarized ^{3}He–HeII mixtures; and (2) the calculation of the thermodynamic properties of these mixtures at temperatures $\leq$0.025 K. We have found that the mean internal energy per unit volume, the pressure, the entropy per unit volume, and the specific heat per unit volume increase with increasing temperature. The mean internal energy per unit volume and the pressure increase with increasing spin polarization; while the entropy per unit volume and the specific heat per unit volume are weakly-dependent on spin polarization.

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